SMOOTH MANIFOLDS FALL 2023 - HOMEWORK 3

SOLUTIONS

Problem 2. Let S_1 and S_2 be level sets of functions $F_1, F_2 : \mathbb{R}^3 \to \mathbb{R}$ at regular values, respectively. Show that $S_1 \cap S_2$ is the level set of a function F (you have to find it), and use this function to show that if for every $x \in S_1 \cap S_2$, $\ker dF_1(x) \neq \ker dF_2(x)$, then $S_1 \cap S_2$ is a 1-manifold.

Solution. Let $y_1, y_2 \in \mathbb{R}$ denote the regular values of F_1 and F_2, respectively, such that $S_i = F_i^{-1}(y_i)$. Define $F(x) = (F_1(x), F_2(x))$, so that F is a function from $\mathbb{R}^3$ to $\mathbb{R}$. We claim that the point $y = (y_1, y_2)$ is a regular value of F. Indeed, observe that $S_1 \cap S_2 = F_1^{-1}(y_1) \cap F_2^{-1}(y_2) = F^{-1}(y)$. Furthermore, note that by assumption $\ker dF_1(x) \neq \ker dF_2(x)$, and both are hyperplanes for every $x \in S_1 \cap S_2$ since y_1 and y_2 are regular values of F_1 and F_2 respectively. We may therefore choose for every $x \in S_1 \cap S_2$, $v \in \ker dF_1(x)$ such that $dF_2(x)v \neq 0$. Hence $dF(x)v$ is a nonzero multiple of e_2. Similary, we may find a vector w such that $dF(x)w$ is a nonzero multiple of e_1. This implies that $dF(x)$ is onto for every $x \in F^{-1}(y)$ and y is a regular value. By the regular value theorem, $S_1 \cap S_2 = F^{-1}(y)$ is a 1-manifold. □

Problem 3. Let $S(x, r)$ denote the sphere of radius r based a a point $x \in \mathbb{R}^3$, and $C(v, r)$ denote the cylinder centered at the line through 0 in the unit vector v of radius r:

$$C(v, r) = \left\{ y \in \mathbb{R}^3 : ||\pi_v(y)|| = r \right\},$$

where $\pi_v : \mathbb{R}^3 \to \langle v \rangle^\perp$ is the orthogonal projection onto the orthogonal complement of v.

(a) Find functions $F_{v,r} : \mathbb{R}^3 \to \mathbb{R}$ and $G_{x,r} : \mathbb{R}^3 \to \mathbb{R}$ such that $C(v, r)$ and $S(x, r)$ are levels sets of F and G respectively.

(b) For which points, vectors and radii are the sphere and cylinder $S(x, r_1)$ and $C(v, r_2)$ transverse? For which are they nontrivially transverse?

Solution. For (a), let $G_{x,r}(y) = |x - y|^2 - r^2$, so that $S(x, r)$ is $G_{x,r}^{-1}(0)$ and $F_{v,r}(y) = |\pi_v(y)|^2 - r^2$,so that $C(v, r)$ is $F_{v,r}^{-1}(0)$. Overse that

$$G_{x,r}(y) = \left(\sum_i {x_i}^2 - 2x_i y_i + {y_i}^2 \right) - r^2 \qquad F_{v,r}(y) = \sum_i y_i^2 - (\langle v, y \rangle)^2 - r^2$$

For (b), observe that the intersection is transverse if and only if $0 \in \mathbb{R}^2$ is a regular value of $H(y) = (G_{x,r}(y), F_{v,r}(y))$. We compute:

$$dH(y) = \begin{pmatrix} dG_{x,r_1}(y) \\ dF_{v,r_2}(y) \end{pmatrix} = \begin{pmatrix} -2x_1 + 2y_1 & -2x_2 + 2y_2 & -2x_3 + 2y_3 \\ 2y_1 - 2v_1 \langle v, y \rangle & 2y_2 - 2v_2 \langle v, y \rangle & 2y_3 - 2v_3 \langle v, y \rangle \end{pmatrix}$$

To verify that the matrix $dH(y)$ is onto, we must check that the rows are nonzero and nonproportional for all $y \in H^{-1}(0, 0)$. The first row is zero if and only if $x_i = y_i$ for every i. In this case, $G_{x,r_1}(y) = -{r_1}^2$, and since we assume $G_{x,r_1} = 0$, $r_1 = 0$. Thus, we must have $r_1 > 0$. The second row is all zeros if and only if $y_i = v_i \langle v, y \rangle$ for all i. Since v is a unit vector, by squaring and summing this equality this occurs if and only if y_i is proportional to v by the Cauchy-Schwartz inequality. If y is proportional to v, then $\pi_y(v) = 0$, and again we conclude that $r_2 = 0$. Thus we must have $r_2 > 0$.

Finally, we check when the rows are linearly independent. Notice that they are scalar multiples if and only if for every i, $y_i - x_i = \lambda(y_i - v_i \langle v, y \rangle)$. If we square and sum each of these terms, we see that $|y - x| = |\lambda| \cdot |y - v \langle v, y \rangle|$. Since we are requiring the we lie in the intersection, $|y - x| = r_1$ and $|y - v \langle v, y \rangle| = r_2$. Hence $|\lambda| = r_1/r_2$. So we get transversality unless $r_2(y-x) = \pm r_1(y - v \langle v, y \rangle) = \pm r_1 \pi_v(y)$ as vectors.

Thus, we lack transversality exactly when $y - x$ and $\pi_v(y)$ are proportional. Now, we write $y = cv + r_2 w$ for some $c \in \mathbb{R}$ and w perpindicular to v. Then since $y - x$ differs in the same direction w with magnitude r_1, it follows that we lack transversality exactly when

$$x = cv + r_2 w \pm r_1 w = cv + (r_2 \pm r_1) w.$$

Equivalently, when $|\pi_v(x)| = |r_1 \pm r_2|$. Finally, observe that the intersection is nontrivial if and only if $|\pi_v(x)| < r_2$ and $|\pi_v(x)| + r_1 > r_2$ or $|\pi_v(x)| \geq r_2$ and $|\pi_v(x)| - r_1 < r_2$. □

Problem 4. Let $M \subset \mathbb{R}^3$ be a compact surface and $E \subset \mathbb{R}^3$ be a plane passing through 0. Show that there exists a plane $E + v$ parallel to E such that $M \cap (E + v)$ is a nonempty union of circles. [*Hint*: If you want to apply Sard's theorem, make sure you read the fine print (ie, pay attention to the difference between a regular value and a non-trivial regular value). The conclusion fails when M is not compact!]

Proof. Let w be a vector perpindicular to E and $p_w(x) = \langle x, w \rangle$, so that p_w is the projection onto $\mathbb{R}$ whose level sets are planes perpindicular to w (ie, sets of the form $E + v$ for some E). Notice that $M \cap (E + v)$ exactly a level set of $p_w|_M$. Thus, it suffices to show that $p_w|_M$ has at least one nontrivial regular value, since any compact 1-manifold is a union of circles. Because $p_w|_M$ is a continuous function, by the intermediate value theorem, the image is either a singleton or closed interval since M is compact. When the image is a closed interval, there exists at least one nontrivial regular value by Sard's Theorem.

Thus, we must show that the image of $p_w|_M$ cannot be a singleton. Notice that if the image was a singleton, then M would be contained in a single preimage, which is a plane. Thus, M is an embedded 2-manifold in a plane. Thus, M is an open submanifold of $\mathbb{R}^2$. Since M is compact, this is a contradiction to connectedness and non-compactness of $\mathbb{R}^2$. □